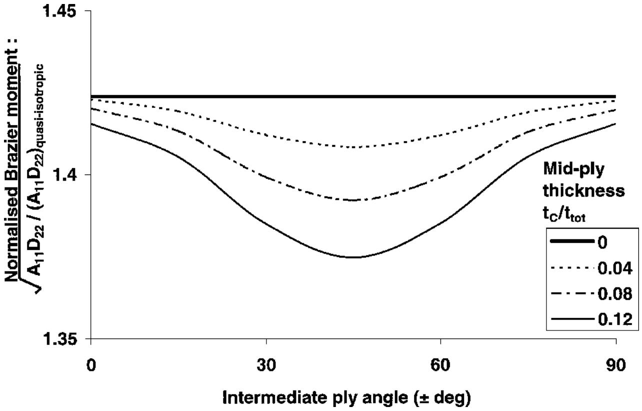


**Fig. 3 Effect of inserting intermediate ply on Brazier[1] moment $(A_{11} \cdot D_{22})^{1/2}$ of a GFRP tube.**

angles $\pm\theta_C$ (Fig. 3), between the middle and outer layers. These layers were introduced into Eq. (5), giving

$$A_{11} \cdot D_{22} = [(\bar{Q}_{11}t)_M + (\bar{Q}_{11}t)_O + (\bar{Q}_{11}t)_C]$$
$$\times [(\bar{Q}_{22}I)_M + (\bar{Q}_{22}I)_O + (\bar{Q}_{22}I)_C]$$

as an expression for the Brazier[1] moment. Once again, the effect of varying ply angles and intermediate layer thickness (for a fixed overall shell thickness) can be found using the Tsai–Pagano[5] relationships. The effect on Brazier[1] moment of varying $t_C$ and $\pm\theta_C$ is plotted in Fig. 3 using the optimum GFRP layup [90, 0-deg]$_s$ and $t_M/t_{\text{tot}}$ of 0.6. As is apparent, the introduction of another, nonorthogonal, layer always reduces the Brazier[1] moment, suggesting that the optimal configuration consists solely of three layers.

## Conclusions

The optimum configuration for a tube under bending is independent of Poisson's ratio and largely invariant with $E_{11}$ and $E_{22}$ for common highly directional composite materials. The optimal ply configuration for typical composite tubes is found to be 90–0–90 deg, with 62% 0-deg plies. Such a configuration increases the maximum bending moment in GFRP tubes by 42% compared to a quasi-isotropic configuration.

## Acknowledgments

We would like to acknowledge the support of the Engineering and Physical Sciences Research Council, United Kingdom, and Westland Helicopters. The second author would like to thank Joachim Grenestedt (Lehigh University) for a discussion on optimal laminates a few years ago.

## References

[1]Brazier, L. G., "The Flexure of Thin Cylindrical Shells and Other 'Thin' Sections," *Late of the Royal Aircraft Establishment*, Repts. and Memoranda 1081 (M.49.), Farnborough, England, U.K., 1926, pp. 1–30.

[2]Vlasov, V. Z., "General Theory of Shells and Its Applications to Engineering," NASA Technical Translation TTF-99, 1964.

[3]Kedward, K. T., "Nonlinear Collapse of Thin-Walled Composite Cylinders Under Flexural Loading," *Proceedings of the 2nd International Conference on Composite Materials*, Toronto, 1978, pp. 353–365.

[4]Harursampath, D., and Hodges, D. H., "Asymptotic Analysis of the Non-Linear Behaviour of Long Anisotropic Tubes," *International Journal of Non-Linear Mechanics*, Vol. 34, No. 2, 1999, pp. 1003–1018.

[5]Tsai, S. W., and Pagano, N. J., "Invariant Properties of Composite Materials," *Composite Materials Workshop*, Technomic, Westport, CT, 1968, pp. 233–253.

A. N. Palazotto
*Associate Editor*

# Thermal Postbuckling of Uniform Columns: A Simple Intuitive Method

G. Venkateswara Rao* and K. Kanaka Raju†
*Vikram Sarabhai Space Center, Trivandrum 695 022, India*

## Nomenclature

| | | |
|---|---|---|
| $A$ | = | area of cross section of the beam |
| $a$ | = | central lateral deflection of the column |
| $E$ | = | Young's modulus |
| $I$ | = | area moment of inertia |
| $L$ | = | length of the beam |
| $P_t$ | = | thermal load |
| $r$ | = | radius of gyration |
| $T$ | = | temperature |
| $U$ | = | strain energy as given by Eq. (1) |
| $u$ | = | axial displacement |
| $W$ | = | work done by the thermal load as given by Eq. (2) |
| $w$ | = | lateral displacement |
| $x$ | = | axial coordinate |
| $\alpha$ | = | coefficient of thermal expansion |
| $\beta_1, \beta_2$ | = | coefficients in Eqs. (3) and (4) |
| $\varepsilon_x$ | = | axial strain including nonlinear terms |
| $\lambda_L$ | = | linear thermal critical load, $(\alpha T L^2/r^2)_L$ |
| $\lambda_{\text{NL}}$ | = | thermal post buckling load, $(\alpha T L^2/r^2)_{\text{NL}}$ |
| $\lambda_{\text{Ta}}$ | = | axial tension parameter as given by Eq. (5) |
| $\psi_x$ | = | curvature |

## Introduction

THE truss- or frame-type structure is widely used in rocket and space structures, for example, interstages, solar sails, etc. These structures are exposed to thermal loading because of aerodynamic or solar heating. Structural elements such as uniform columns are the basic components of these structures, and the prediction of their postbuckling behavior is an important design input. Furthermore, their postbuckling strength can be effectively used in achieving an optimum (minimum mass) design of these structural elements.

Postbuckling behavior of columns subjected to mechanical loads was discussed by Dym[1] and Thompson and Hunt[2] using a differential equation approach. Although this approach gives exact solutions for some simple structural configurations, it is not easy to obtain solutions when complex geometries, loads, and boundary conditions are involved. As such, one has to resort to numerical methods to obtain solutions for such structural configurations. A numerical method, such as the versatile finite element method, and a continuum method, such as the Rayleigh–Ritz method, were used to investigate the thermal postbuckling behavior of uniform columns with immovable ends by Rao and Raju.[3] However, these methods are relatively tedious. Even though the finite element formulation is highly versatile, one has to idealize the column with a large number of elements, and the solution involves a large number of iterations, to achieve a desired degree of accuracy in evaluating $\lambda_{\text{NL}}$.

In the present Note, a simple, intuitive method is proposed to predict the thermal postbuckling behavior of uniform columns with different boundary conditions, after briefly describing the Rayleigh–Ritz formulation.

Received 18 November 2001; revision received 8 April 2002; accepted for publication 25 June 2002. 

*Group Director, Structural Engineering Group; gv_rao@vssc.org.

†Head, Software Development Section, Structural Design and Analysis Division.

**Table 1 Values of $\lambda_L$ and $\gamma$ of columns with different boundary conditions**

| Boundary conditions | $\lambda_L$ Present study | $\lambda_L$ Finite element method[a] | $\lambda_L$ Rayleigh–Ritz method[a] | $\gamma$ Present study | $\gamma$ Finite element method[a] | $\gamma$ Rayleigh–Ritz method[a] |
|---|---|---|---|---|---|---|
| Pinned–pinned | $\pi^2$ | 9.8699 | $\pi^2$ | 0.25 | 0.25 | 0.25 |
| Clamped–clamped | $4\pi^2$ | 39.4985 | —— | 0.0625 | 0.0624 | —— |
| Pinned–clamped | 20.3474 | 20.2322 | —— | 0.1504 | 0.1480 | —— |

[a]See Ref. 3.

## Rayleigh–Ritz Method

In the Rayleigh–Ritz method, the total potential energy $(U - W)$ is minimized with respect to the undetermined coefficients of the assumed admissible functions. $U$ is the strain energy given by

$$U = \frac{EA}{2}\int_0^L (\varepsilon_x)^2\,dx + \frac{EI}{2}\int_0^L (\psi_x)^2\,dx \tag{1}$$

and $W$ is the work done by the thermal load $P_t (= E\alpha\, TA)$ given by

$$W = \frac{P_t}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \tag{2}$$

For the case of a simply supported column with immovable ends, the admissible functions for the lateral displacement $w$ and the axial displacement $u$ that satisfy the geometric boundary conditions are

$$w = \beta_1 \sin(\pi x/L) \tag{3}$$

$$u = \beta_2 \sin(2\pi x/L) \tag{4}$$

Minimizing the total potential energy with respect to undetermined coefficients $\beta_1$ and $\beta_2$, we get a system of nonlinear algebraic equations that can be solved for $\lambda_{NL}/\lambda_L$. The main difficulty of this method is arriving at a suitable admissible function for $u$ for the boundary conditions other than the simply supported case.

## Present Intuitive Method

The present intuitive method is developed based on the physics of the problem. If the column with immovable ends undergoes large lateral deformations, the axial tension parameter $\lambda_{Ta} (= T_a L^2/EI$, where $T_a$ is the axial tension developed in the column) is given by[4]

$$\lambda_{Ta} = \frac{L}{2r^2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \tag{5}$$

If the column is subjected to a uniform temperature rise $T$ from the stress free condition, an axial compressive force is developed in the column. This compressive force reaches a critical value, the linear critical thermal load $\lambda_L$, when it balances the linear elastic forces. However, in the postbuckling range, the compressive force developed also has to balance the axial tension developed due to large deformations in addition to the linear elastic forces. Thus, the thermal postbuckling load $\lambda_{NL}$ can be thought of as

$$\lambda_{NL} = \lambda_L + \lambda_{Ta} \tag{6}$$

or

$$\lambda_{NL}/\lambda_L = 1 + \lambda_{Ta}/\lambda_L \tag{7}$$

Equation (7) gives the value of $\lambda_{NL}/\lambda_L$ for a specified value of $a/r$.

## Results and Discussion

When the intuitive method described in the preceding section is used, the postbuckling behavior of uniform columns with three types of boundary conditions are computed in terms of $\lambda_{NL}/\lambda_L$. Figure 1 shows a uniform column with immovable ends, that is, $u = 0$ at the ends, with a part of the deformed shape at the central region. The deformed shape at the ends is not shown because it varies with the boundary conditions at the ends.

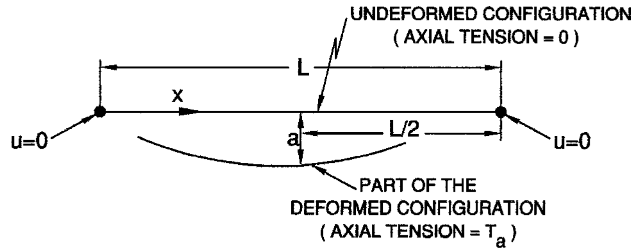


**Fig. 1 Uniform column with immovable ends in axial direction.**

### Pinned–Pinned Column

An exact displacement distribution taken from Timoshenko and Gere[5] for this boundary condition is

$$w = a \sin(\pi x/L) \tag{8}$$

$$\lambda_L = \pi^2 \tag{9}$$

Then $\lambda_{Ta}$ obtained from Eq. (5) is

$$\lambda_{Ta} = (\pi^2/4)(a^2/r^2) \tag{10}$$

From Eq. (7),

$$\lambda_{NL}/\lambda_L = 1 + 0.25(a^2/r^2) \tag{11}$$

### Clamped–Clamped Column

The exact admissible function for this case is[5]

$$w = (a/2)[1 - \cos(2\pi x/L)] \tag{12}$$

$$\lambda_L = 4\pi^2 \tag{13}$$

Again for the assumed displacement given by Eq. (12), from Eq. (5)

$$\lambda_{Ta} = (\pi^2/4)(a^2/r^2) \tag{14}$$

and, hence, from Eq. (7)

$$\lambda_{NL}/\lambda_L = 1 + 0.0625(a^2/r^2) \tag{15}$$

### Pinned–Clamped Column

For this boundary condition, an exact admissible displacement distribution is not readily available, and a two term admissible function is derived by the authors as

$$w = a_1(\xi/2 - 3\xi^3/2 + \xi^4) + a_2(\xi - 2\xi^3 + \xi^5) \tag{16}$$

where

$$\xi = x/L \tag{17}$$

which satisfies all four of the boundary conditions of the clamped–pinned column. When the Rayleigh–Ritz method is used, as described in the preceding section and without considering the axial energy terms in $U$, the critical load parameter $\lambda_{cr}$ and the corresponding axial tension parameter are obtained as

$$\lambda_L = 20.3474 \tag{18}$$

$$\lambda_{Ta} = 3.06135 \tag{19}$$

The eigenvector $[a1 \;\; a2]^T$, where $[\,]^T$ denotes transpose, is calculated using the eigenvalue $\lambda_L$, such that the central displacement is unity.

Note here that no assumption need be made on the axial displacement $u$ because this is not required for the present intuitive method. From Eq. (7), we obtain

$$\lambda_{\mathrm{NL}}/\lambda_L = 1 + 0.1504(a^2/r^2) \tag{20}$$

In all of the cases, $\lambda_{\mathrm{NL}}/\lambda_L$, in general, can be represented for the three boundary conditions considered by

$$\lambda_{\mathrm{NL}}/\lambda_L = 1 + \gamma(a^2/r^2) \tag{21}$$

Values of $\gamma$ obtained from the present simple intuitive method are tabulated along with those obtained from the finite element method and the Rayleigh–Ritz method (see Ref. 3) in Table 1 and can be seen to be in excellent agreement with those given by Rao and Raju.[3]

## Conclusions

A simple, intuitive method is proposed to predict the thermal postbuckling behavior of uniform columns, with immovable ends. This method is applied to obtain the thermal postbuckling load of columns with different boundary conditions. The present results are in excellent agreement with those available in the literature, showing the efficacy of the proposed method.

Note that the accuracy of the proposed method depends on the accuracy of the assumed function for the lateral displacement just like the Rayleigh–Ritz method. The proposed method gives an exact solution if the assumed function for the lateral displacment is exact; otherwise, the method gives an approximate solution. This aspect can be clearly seen from the numerical results presented in this Note.

## References

[1]Dym, C. L., *Stability Theory and Its Applications to Structural Mechanics*, Noordhoff International, Leyden, The Netherlands, 1974.

[2]Thompson, J. M. T., and Hunt, G. W., *A General Theory of Elastic Stability*, Wiley, London, 1973.

[3]Rao, G. V., and Raju, K. K., "Thermal Postbuckling of Columns," *AIAA Journal*, Vol. 22, No. 6, 1984, pp. 850, 851.

[4]Woinowsky-Krieger, S., "The Effect of an Axial Force on the Vibration of Hinged Bars," *Journal of Applied Mechanics*, Vol. 17, March 1950, pp. 35, 36.

[5]Timoshenko, S. P., and Gere, J. M., *Theory of Elastic Stability*, McGraw–Hill, New York, 1961.

E. R. Johnson
*Associate Editor*